# Exploring GR effects in Newtonian physics

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Bohdan Paczyński pioneered pseudo-Newtonian calculations that captured essential qualitative effects of general relativity in accretion disk theory. After reviewing the historical successes of this approach, I will discuss analogues of general-relativistic effects which are present in the strictly Newtonian physics of orbital motion around Maclaurin spheroids. Oscillations modes of accretion disks that are familiar from relativistic diskoseismology seem to be present also in strictly Newtonian physics.

## 1 Introduction

It is a great honour to have been invited to give a talk in a special session of the Polish Astronomical Society (PTA) in which the Bohdan Paczyński Medal of the PTA has been awarded to its first recipient, Professor Martin Rees, Baron Rees of Ludlow. In this contribution, as a tribute to Bohdan Paczyński, I will briefly review the applications of the famous Newtonian pseudo-potential introduced by Paczyński (in what is his third most cited paper, with 844 citations at the time of the PTA session: Paczyński & Wiita, 1980). I will then discuss current research on circular orbits in the strictly Newtonian potential of Maclaurin spheroids which shows analogies with relativistic results.

## 2 Paczyński's Newtonian pseudo-potential

The theory of black hole accretion disks started 40 years ago with the papers of Shakura & Sunyaev (1973) and Pringle et al. (1973). Interestingly, while the first paper contains the equations of motion relevant to disk-like flows, they are Newtonian. The main qualitative effect of Einstein's general relativity (GR) was inserted into the problem as a boundary condition, specifically that of zero torque at the marginally stable orbit, $r_{\rm ms} = 6M(G/c^2)$ in the Schwarzschild metric, today usually called the innermost stable circular orbit (ISCO). The relativistic version of the equations and their solution in the Kerr metric presented in Novikov & Thorne (1973) used the same inner boundary condition. This led to unphysical infinities at the inner edge of the disk. It took nearly a decade to correctly describe the flow through the inner edge of the disk, and the solution was only made possible by a crucial insight of Bohdan Paczyński, who proposed a Newtonian model of GR.

It has to be understood that three decades ago computers were not sufficiently developed to tackle the difficult problem of accretion disks in GR. While numerical relativistic hydrodynamics codes existed, they were incapable of handling two-dimensional problems in a reasonable CPU time. Analytical solution of flow through the inner boundary of an accretion disk also proved too challenging, although some insight into the problem was achieved in a Schwarzschild metric calculation by Stoeger (1980).

Stationary toroidal configurations[1] of non-gravitating fluid in axisymmetric GR metrics were presented in a paper by Paczyński's junior collaborators, Abramowicz et al. (1978). It allowed to model accretion flow through the inner edge of the disk, a so-called Sikora's beak[2], in a manner analogous to Roche-lobe overflow in binary systems. However, the only feasible approach to modelling the inner structure of a thin accretion disk was through a numerical solution of the equations of (viscous Shakura & Sunyaev, 1973, alpha disk) motion.

The problem that needed to be solved was how to reconcile Newtonian gravity with the presence of the marginally stable orbit. This was accomplished brilliantly by considering circular orbits in a spherical pseudo-potential (Paczyński & Wiita, 1980)

$$V(r) = -\frac{GM}{r - r_{\rm G}}, \tag{1}$$

with $r_{\rm G} = 2GM/c^2$ the Schwarzschild radius. This led to the appearance, in a purely Newtonian calculation, of a marginally stable orbit (at the correct radius! $r_{\rm ms} = 3r_{\rm G}$). It was then possible to show that radial flow could be accounted for (Paczyński & Bisnovatyi-Kogan, 1981), while the correct inner boundary condition is that of transonic flow (Muchotrzeb & Paczyński, 1982). And thus opened the floodgates of accretion disk research.[3]

It is worth noting that the pseudo-potential given in eq. (1), and its variants[4], are used even today for modeling phenomena as diverse as binary coalescence of black holes and neutron/quark stars (Kluźniak & Lee, 1998, 2002), accretion disk oscillations (Horak & Lai, 2013), and gamma-ray bursts arising in collapsars (Lopez-Camara et al., 2009). Of course, ray-bending cannot be modelled in this way, so for this and other reasons photon spectra obtained in such calculations have to be taken with a grain of salt.

## 3 Kepler's law and epicyclic frequencies

The presence of a marginally stable orbit allows the trapping of acoustic-inertial modes of oscillation in the inner parts of an accretion disk (Kato & Fukue, 1980; Perez et al., 1997). Interestingly, Kepler's law $P^2 \propto R^3$ has an identical form in Newton's $1/r^2$ gravity and in the Schwarzschild metric, the Keplerian frequency being $\Omega_{\rm K}(r) = GM/r^3$. For the spherically symmetric case, the vertical epicyclic frequency (related to the restoring force when a perturbation out of the orbital plane occurs) is identical to $\Omega_{\rm K}(r)$. However the radial epicyclic frequency, which is related to the stability of circular orbits has a different form in GR than in the Newtonian case. Related to this, and to the instability of orbits for all $r < r_{\rm ms}$, is the difference in angular momentum: in the Newtonian $1/r$ potential $l_{\rm K}^2 = GMr$ is increasing with $r$, thus satisfying Rayleigh's criterion for stability, while in the Schwarzschild case $l_{\rm K}^2 = GMr/(1 - r_{\rm G}/r)^2$ has a minimum at $r = r_{\rm ms}$ and its radial derivative is negative for $r < r_{\rm ms}$.

[1] They were called "Polish doughnuts" by Professor Rees.

[2] This is a play on Marek Sikora's name, which in Polish signifies a titmouse.

[3] Today, it is an easy matter to solve for the structure of radiative accretion disks in the Kerr metric, now that the proper boundary condition is known (e.g., Sądowski et al., 2011).

[4] A potential co-developed by the current author, $V(r) = (c^2/6)[1 - \exp(3r_{\rm G}/r)]$ (Kluźniak & Lee, 2002) correctly models the ratio $(1 - r_{\rm ms}/r)^{1/2}$ of the epicyclic frequency to the orbital frequency in Schwarzschild geometry.

Inclusion of rotation makes things more interesting. The degeneracy between the orbital and vertical epicyclic frequency is broken: the larger the spin rate of the central object, the greater the difference between the vertical epicyclic frequency and the orbital one. Because of frame dragging, in Kerr geometry the values of the three frequencies in the equatorial plane now depend on whether the orbits are in the same sense as that of the rotation of the black hole, or in the opposite sense. For prograde orbits the vertical epicyclic frequency is smaller than the orbital one, for retrograde orbits, the vertical epicyclic frequency is larger. This has very interesting consequences for accretion-disk oscillation modes. As shown by Silbergleit et al. (2001), the (corrugation) *c*-mode corresponding to the Lense-Thirring precession is present only for disks in prograde motion. In both cases the radial epicycle is lower than the orbital frequency. In Kerr geometry the maximum value of the radial epicyclic frequency increases with the spin of the black hole for prograde orbits, while its zero, i.e., the location of the marginally stable orbit moves to smaller radii (closer to the black hole).

In Newtonian gravity rotation of the central body also removes the degeneracy between the epicyclic frequencies and the orbital one. However, as there is no frame dragging, there is no difference between prograde and retrograde orbits. A sizable octupole moment for an oblate body, with its $-1/r^3$ contribution to the potential, can overcome the centrifugal barrier and thus destabilize orbits within a certain radius, at which a marginally stable orbit will be located. This was shown for the classical Maclaurin spheroids by Kluźniak et al. (2001); Amsterdamski et al. (2002). Kluźniak & Rosińska (2013) give analytic expressions for the orbital and epicyclic frequencies as a function of the ellipticity of the spheroid and trapped modes in accretion disks around Maclaurin spheroids are discussed in Khanna & Strzelecka et al. (2014).

## 4 Maclaurin spheroids and quark stars

Epicyclic and orbital frequencies are of great interest in the context of disk oscillations (Kato & Fukue, 1980; Wagoner et al., 2001; Silbergleit et al., 2001) and the observed millisecond oscillations (kHz QPOs) in X-ray binaries (van der Klis M., 2000). We show these frequencies for a $1.4M_\odot$ quark star in Fig. 1. The calculations have been performed for an MIT bag model with the equation of state $P = (\rho - \rho_0)c^2/3$, with $\rho_0 = 4.2785 \times 10^{14}\mathrm{g/cm}^3$. At moderate rotation rates, e.g. at 600 Hz, the frequencies resemble those in the Kerr metric: the vertical epicyclic frequency (dashed line) is just below the orbital frequency (thin solid line), while the radial epicyclic frequency (dotted line) is slightly deformed from its Schwarzschild value, with the marginally stable orbit just inside $6GM/c^2$. However, a rapidly rotating model (at 1165 Hz) shows a qualitatively different behavior, the value of the radial epicyclic frequency is diminished, with the marginally stable orbit pushed out to nearly $8GM/c^2$, while the vertical epicyclic frequency is now *larger* than the orbital frequency. This is somewhat reminiscent of retrograde orbits in the Kerr metric, but here we are discussing prograde motion only. An explanation for this unusual behaviour can be found in the oblateness of the rotating configuration.

We can compare (Gondek-Rosińska et al., 2014) rotating quark star models computed with a fully general-relativistic code to the purely Newtonian results for Maclaurin spheroids (Kluźniak & Rosińska, 2013). Quark stars are not as centrally condensed as neutron stars (and are self-bound), so low-mass quark stars are fairly homogeneous. One expects uniform density Maclaurin spheroids to provide a good model for their

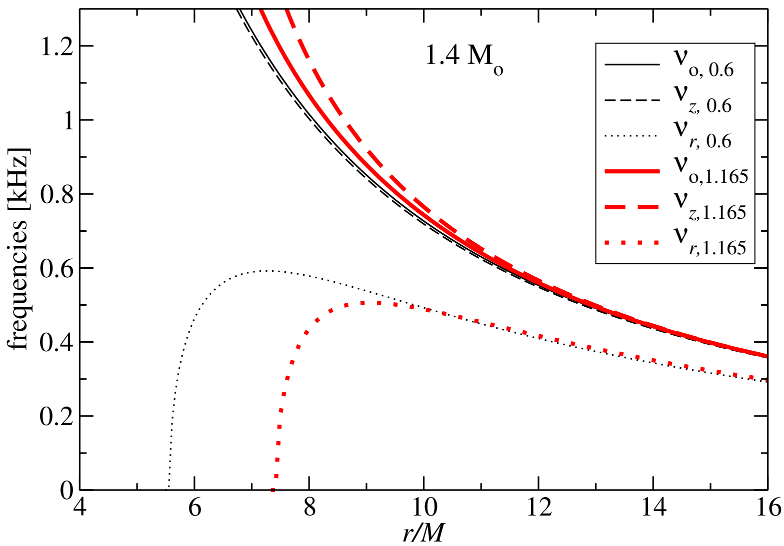


Fig. 1: The orbital and epicyclic frequencies for 1.4 solar mass quark stars spinning at 600Hz (black) and 1165 Hz (red). The figure is from Gondek-Rosińska et al. (2014).

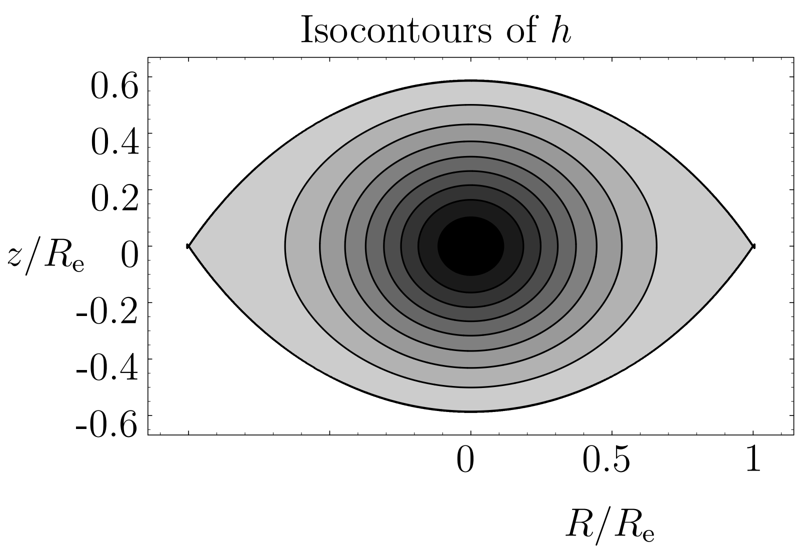


Fig. 2: Density contours of a rotating quark star.

external gravitational potential. Fig. 2 illustrates the density contours of a rapidly spinning quark star configuration, while Fig. 3 provides a comparison of the analytic Maclaurin spheroid results (solid curves) with numerical calculations for a $0.001M_\odot$ quark star. The agreement is excellent, which points to effects of oblateness being responsible for the qualitative behaviour of the epicyclic frequencies of rapidly rotating compact stars.

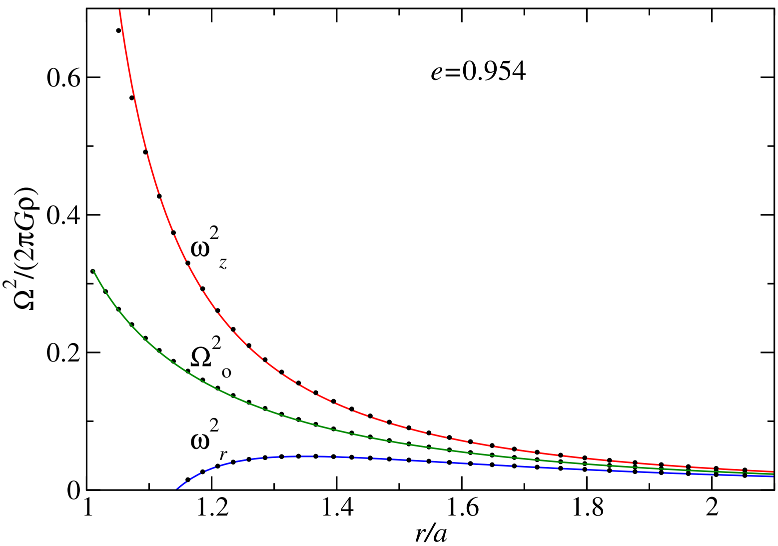


Fig. 3: Epicyclic and orbital frequencies squared of a highly oblate $0.001M_\odot$ quark star (dots) and of a corresponding Maclaurin spheroid (solid lines). The radius is scaled with the equatorial radius of the star, and the frequencies with $\sqrt{2\pi G\rho}$. The figure is from Gondek-Rosińska et al. (2014).

Surprisingly, not only can qualitative features of GR be modeled with a Newtonian pseudo-potential (Paczyński & Wiita, 1980), but some of these seemingly GR features (the presence of a marginally stable orbit, splitting of epicyclic and orbital frequencies) are sometimes of Newtonian origin (Kluźniak & Rosińska, 2013).

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