Lecture I. Dark Matter

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This article summarizes the first of a series of lectures delivered at the Cosmology School "Introduction to Cosmology". It reviews the problem of undetected mass or invisible matter in cosmology in its historical context.

1 Introduction

Virginia Trimble (Trimble, 1987) in her excellent review of Dark Matter set out part of the problem at the beginning of the historical introduction. The observations (Bessel, 1844) by Friedrick Wilhelm Bessel of Sirius and Procyon indicated the presence of a companion, invisible at the time and discovered only later by Alvan G. Clark (Bond, 1862). Clark discovered the white dwarf Sirius B thanks to a larger instrument; that is a step forward in technology allowed the discovery and to settle the matter. This happened quite often in astronomy and in other disciplines, the detected or hypothesized "invisible" becomes "visible". Our tremendous improvement in technology and theoretical capabilities allows us to set extremely accurate and sophisticated boundaries to the problems and give us confidence that we are getting closer and closer to the solution.

These lectures on Dark Matter (DM) will deal with the problem of undetected mass or invisible matter, its mass estimate at various scale lengths and the search for it. I will try to indicate the process by which at some point we realized we had a problem, in the sense that we became convinced that we likely are dealing with unknown new non-baryonic particles or eventually with the need for new physics. It will be difficult and rather subjective to state with certainty when and who first realized that we could not explain the observations by invoking only baryonic matter, to a large extent the issue is however of rather minor importance even if historically relevant. Certainly nucleosynthesis, the formation of light elements, set a very firm first limit on the amount of baryonic matter we have in the Universe. The path of thoughts, the way I witnessed it and the way I interpreted it by scanning the literature to my disposal, should reflect the evolution of the process of knowledge and help and stimulate the students toward the challenges presented by new ambitious goals.

We will use kind of an historical approach detailing in some cases with the observations and the related analysis in order to gain a deeper understanding. I will mainly refer to the work based on the gathering of new data and on original theoretical developments aimed to increase our understanding and explain the observational evidence. Indeed the main tracks of research are based on the understanding of the problem, planning and carrying out the observations and the experiments on the one hand and developing theories and tools on the other hand. These approaches will deepen our understanding and lead to the advancement of knowledge the way we know it today. We will gradually approach the problem from small scales to larger scales and understand whether or not we have significant differences. This writing is not a review paper and this will in part justify the many scientists I will not refer to and the many papers I may have missed. It would have been impossible anyway to refer all of the very significant papers due to the huge amount of excellent work done in this field in the past years. I apologize for the work I eventually missed and I would appreciate comments and criticism, in addition to information on missed literature. This would help me in improving the text.

The understanding of Dark Matter is fundamental to our knowledge in physics. We realize we are in a very odd situation, indeed we are in a situation that is not acceptable. We live in a Universe like a few ghosts would live in a large forest of which they are no part, their essence differs. The feeling of physicists must be even worse of the contradictions that many felt at the time when we needed the ether. A Deus ex machina that would help in some cases but that finally led to many contradictions [see the translation of the seminar Einstein gave in Leiden (Einstein, 1922). To some extent the Dark Matter may seems to have a similar role with the fundamental difference that it manifests itself through a fundamental force: gravitation! And, so far, it works! on the other hand we need to detect it or, also a fascinating possibility, develop a new theory of physics.

Limit m_{pg}	Nebulae	Scurce	Interval	Nebulae*
14.5	1	Measured magnitudes	14.0–14.5	1
15.0	4	Measured magnitudes	14.5–15.0	3
15.5	17	Measured magnitudes	15.0–15.5	14
16.2	60	Measured magnitudes	15.5–16.0	29
16.5	80	10-inch focal	16.0–16.5	34
17.2	140	100-inch extra-focal	16.5–17.0	43
17.3	150	60-inch extra-focal	17.0–17.5	38
18.1	202	100-inch extra-focal	17.5–18.0	34
19.2	255	36-inch focal†	18.0–18.5	29
19.5	266	100-inch focal	18.5–19.0	23
20.5	263	100-inch focal	19.0–19.5	15
			19.5–20.0	2
			20.0–20.5	0

* Numbers were read from a smooth curve representing the data in the first two columns.

† Photograph with the Crossley reflector, available through the courtesy of the director of the Lick Observatory.

Fig. 1: Tab. VI from Hubble & Humason (1931): Distribution of magnitudes of 'nebulae' in the Coma cluster.

In the first lecture after a brief introduction I deal with the search of the gravitational mass in the solar neighborhood and look into the measurements of the gravitational mass of the Milky Way. Moving toward larger scale size we discuss the mass in binaries and larger system of galaxies giving some details on the estimates via strong and weak lensing. Since the work by Zwicky (1933) clusters of galaxies played a leading role in estimating the discrepancy between the mass estimated using the light of galaxies and the dynamical mass. The third lecture deals with clusters of galaxies. At this point we have evidence on various scales that in the framework of the theory we know (Newton & Einstein) the visible matter alone is incapable of generating the gravitational force we need to explain the dynamics we observe. In these lectures I do not touch on the evidence related to the Large Scale Structure and the growth of perturbations, the matter will be discussed eventually in the near future. The nature of the invisible hypothesized matter is at present completely unknown. As expected the search for its identity and the related theoretical development is huge. In the fourth lecture, also because of its interest in relation to the thermal evolution of the early Universe, we will briefly touch upon the WIMP, freeze out time, abundance and cross section. For axions we refer, in addition to the copious literature on this matter, to the related excellent chapter in the book by Kolb & Turner (1990). The large development of experiment and receivers and the progress made in the theory are beyond the scope of these lectures, we will briefly mention however the DAMA experiment and the Modified Newtonian Dynamics (MOND).

As a point of reference the parameters measured by the Planck collaboration on very large scales are: $\Omega_{\rm b}h^2 = 0.0224 \pm 0.001$ and $\Omega_{\rm m}h^2 = 0.14 \pm 0.002$. With $\Omega_{\rm b}/\Omega_{\rm m} = 0.16$: most of the matter in the Universe is non-baryonic.

2 Lecture I

F. Zwicky published the well known paper "Die Rotverschiebung von Extragalaktischen Nebeln" (Zwicky, 1933) where he discusses the observed redshifts of the galaxies, the correlation velocity distance and in particular the velocity dispersion of the Coma Cluster of galaxies reaching the following conclusion:

> "Falls sich dies bewahrheiten sollte, wurde sich also überraschende Resultat ergeben, dass dunkle Materie in sehr grösserer Dichte vorhanden ist als leuchtende Materie."

> "If this is true, it also would be true the surprising conclusion that dark matter is present in much bigger density than luminous matter."

A couple of years earlier Hubble & Humason (1931) published the very detailed work on extragalactic nebulae, in which they not only derived the velocity distance relation but also gave a detailed account of the photometry available at the time and included the catalogue of velocities and magnitudes of extragalactic nebulae, (see example in Fig. 1).

The Coma cluster, thanks to the work of Wolf (1901), Curtis (1918) and Duncan (1923), appeared as a very symmetric and regular cluster when compared to the rather irregular and extended Virgo cluster. In addition, and due to the fact that morphologically the cluster presents a very regular form, photometric observations were obtained for a large number of galaxies so that the visible content and the luminosity function, see Fig. 2, were rather well under control. At the time only 8 redshifts were known[1] and are listed in Tab. 2.

NGC 4865[2] as stated by Hubble & Humason (1931) and by Zwicky (1933) being likely an interloper. The reasoning is very simple. If we have about 800 galaxies (from counts given in Hubble and Humason) whose mass is about 109 $M_{\odot}$ the visible matter in the cluster is about 1.6×10^{45} g. The potential energy then is $\Omega =$

[1] The present values do not differ much from those used by Zwicky.

[2] See however more recent work on the distribution of redshifts of the Coma cluster as part of the Coma – A 1367 Supercluster.

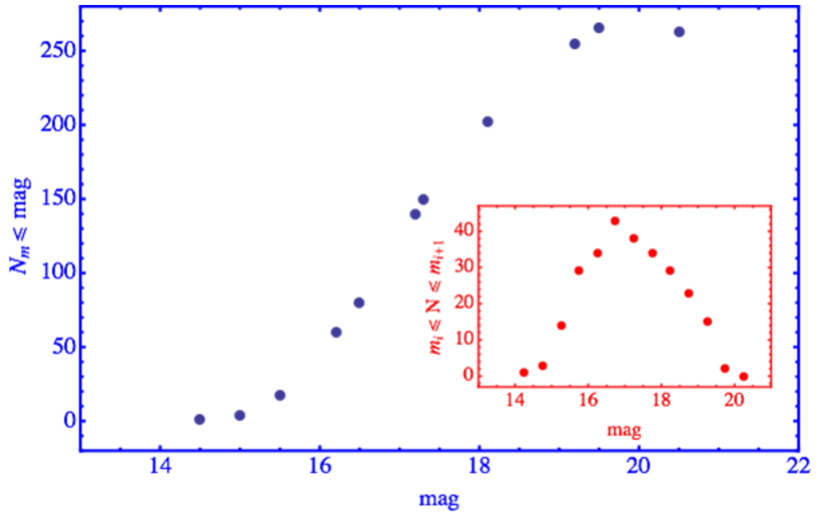

Fig. 2: Cumulative and differential counts as given by Hubble and Humason (1931). The differential counts in the inset have been computed from the smooth curve fitting the cumulative counts.

NGC	**Redshift km s^{-1}**	**NGC**	**Redshift km s^{-1}**
4853	7600	4872	6600
4860	7900	4874	6900
4884	6700	4881	7000
4865	5100	4985	8500

Tab. 1: Objects and their redshifts known at the early thirties.

$-\frac{3}{5}G\frac{M}{R} = -6.4 \times 10^{13}$ cm^2 s^{-2} and the expected velocity dispersion, following the steady state assumption and the application of the viral theorem ($2E_{\mathrm{kin}} + \Omega = 0$), must be $\sigma = \sqrt{2E_{\mathrm{kin}} = -\Omega} = 80$ km s^{-1}. However the observed velocity dispersion is about a factor 10 higher. Either we had a problem in the interpretation of redshift[3], or the cluster is not in equilibrium (expanding) invalidating therefore the application of the viral theorem, or we had a large amount of unseen matter.

Zwicky did not use the Mass to Luminosity ratio in spite of the fact that luminosities were computed by Hubble & Humason (1931). These authors derive a distance modulus for the cluster $< m > -M = 30.8$ by which we would derive an extremely low luminosity and derive for the cluster as a whole a mass to luminosity ratio far from any reasonable value. By using $(m - M)_{\mathrm{Coma}} = 34.4$ as derived by Rood & Williams (1993) and the Luminosity Function measured by Hubble & Humason (1931) we get $(M/L)_{\mathrm{Coma}} = 360(M/L)_{\odot}$.

Since 1925 Zwicky was at Caltech (Pasadena, California), so that it seems strange that such an innovative paper had been published in German in the Swiss *Helvetica Physics Acta* (Zwicky, 1933) rather than in the Ap.J. Perhaps he published on a

[3]See *Morphological Astronomy* (Zwicky, 1957) for some ideas Zwicky had on redshift. The Hubble constant was $H_0 = 550$ km s^{-1} Mpc^{-1}.

Swiss journal in German to spread in Europe the results related to the work by Hubble and Humason (Europe at that time was practically not much involved in Extragalactic Astronomy) and eventually to avoid to be rejected by a journal. At the same time he would avoid to give away ideas before publication (this is only my guess). More recently these inconvenient worked out also the other way around since publishing in a local or Observatory Bulletin the work would be published before the competitors would get the paper printed by a regular journal that obviously had longer technical times. Improper things now and then happened, and happen, also in science.

To conclude and to stress how far ahead of time Zwicky was, I reproduce from that paper a fundamental passage related to his cosmological discussion:

> "Einer Expansion von 500 km/sek pro Million Parseks entspricht nach EINSTEIN und DE SITTER eine mittlere Dichte $\rho \cong 10^{-28}$gr/cm^3. Aus den Beobachtungen an selbstleuchtender Materie schätzt HUBBLE $\rho \sim 10^{-31}$gr/cm^3. Es ist natürlich möglich, dass leuchtende plus dunkle (kalte) Materie zusammengenommen eine bedeutend höhere Dichte ergeben, und der Wert $\rho \sim 10^{-28}$gr/cm^3 erscheint daher nicht unvernünftig."

> "An expansion of 500 km/sec per million parsecs corresponds in the EINSTEIN and DE SITTER model to an average density $\rho \cong 10^{-28}$gr/cm^3. From the observations of self-luminous matter HUBBLE estimates $\rho \sim 10^{-31}$gr/cm^3. It is of course possible that glowing plus dark (cold) matter taken together result in a significantly higher density, and the value $\rho \sim 10^{-28}$gr/cm^3 not unreasonable[4]."

$$\frac{3H_0^2}{8\pi G} = 4.7 \times 10^{-28} \mathrm{g\ cm}^{-3}.$$

As we will see later the modern values for the Coma Cluster are, from Colless and Dunn (1999), $< cz >= 6853\ \mathrm{km\,s^{-1}}$, $\sigma_{\mathrm{cz}} = 1082\ \mathrm{km\,s^{-1}}$, $M = 0.9 \times 10^{15}\ h^{-1} M_{\odot}$.

In the Netherlands a major project was started by Kapteyn (1922) with the goal of estimating the distribution of stars in the Milky Way and to understand the structure of the Galaxy itself. These studies were later continued mainly by F.H. Oort on new theoretical grounds and represent a milestone in the understanding of the morphology and dynamics of the Milky Way. Here again the assumption is made that the Galaxy is in a steady state so that the kinetic energy of the stars is counterbalanced by the mass. The analysis by Oort measures a total density of matter near the Sun equal to $6.3 \times 10^{-24}\mathrm{g\,cm^{-3}}$ or 0.092 solar masses per cubic parsec while the total of the stars down to visual absolute magnitude 13.5 is found to be 0.038 solar masses per parsec cube. He also finds an "indication that the invisible mass is more strongly concentrated to the galactic plane than that of the visible stars". Indeed one of the purposes of the work "was the derivation of an accurate value for the total amount of mass, including dark matter, corresponding to a unit of luminosity in the surrounding of the Sun". Here the scale length is of about 1 kpc, that is about a factor 103 smaller than the scale length considered above for the Coma Cluster.

[4] He doesn't exclude the possibility of a closed Universe with $\Omega_{\mathrm{m}} = 1$.

The study of estimating the mass present in the solar neighborhood is fascinating both from the theoretical point of view and for the observations and their evolution from Kapteyn (fl. 1851–1922) to the present. However the opening of the modern scenario and the consciousness that the dynamical matter was somewhat larger than the visible matter (originally the goal was to search for faint stellar population and account properly for the dust absorption) came in part in 1932 with the fundamental paper by Oort (1932). The milestones are the papers, following also the illuminating work by Jeans (1915), written later by Hill (1960), Oort (1960, 1965a), Bahcall & Soneira (1980, 1984), Kuijken & Gilmore (1989a,b,c); Kuijken (1991) and recently by Zhang et al. (2013) and a few others. The concept is rather simple and based on the assumption that the solar neighborhood stars are part of a galaxy in dynamical equilibrium[5], indeed a collisionless system in dynamical equilibrium[6]. Encounters are rare and there has been no time to set a statistical equilibrium via collisions.

Let's first look at the dynamics in a very simple way. For the Earth atmosphere we can write $dP = -\frac{\rho A dh g}{A} = \rho g dh$ and $dP = \frac{kT}{m} d\rho$ from the two equations we have: $\frac{d\rho}{\rho} = -\frac{m}{kT} g dh$, the running of the density as a function of hight and of the temperature (velocity dispersion). The reasoning by Kapteyn (1922) was exactly this. If I call $< Z^2 >$ the mean square velocity of stars in the z direction where z is the axis perpendicular to the plane of the Galaxy this is related to the force perpendicular to the galactic plane by the relation $K^{\mathrm{z}} =< Z^2 > \frac{d\rho(z)}{\rho}$ where $\rho(z)$ is the density distribution of such stars, see Oort (1932) and Hill (1960) for a lucid discussion of the theory, observations and data analysis. The basic assumption in the analysis is that we are dealing, as stated above, with a system in dynamical equilibrium where the low density of the field characterize a collisionless system and the large number of stars move under the influence of a smooth potential $\Phi(x, t)$. Such system, characterized by a distribution function of seven variables $f(\overline{x}, \overline{\upsilon\nu}, t)$ is then governed by the Boltzmann equation. The solution is rather difficult. However, following Jeans (see also Binney & Tremaine, 1987), we can simplify considerably the problem. By defining a spatial density (number counts) as $\upsilon\nu(\overline{x}) \equiv \int f d^3\upsilon$ and a mean stellar velocity $\overline{\upsilon_i} \equiv \frac{1}{\nu} \int f \upsilon_i d^3\overline{\upsilon}$ in a steady state system, $\frac{\partial}{\partial t}(\Phi) = 0$. Using cylindrical coordinates (Fig. 3) and eliminating small terms in an axis symmetric system that is also symmetric respect a plane (the galactic plane) we derive equation (1) below. Using the Poisson equation (2) and combining it with equation (1) we derive equation (3). Equation (4) by integrating over z:

$$\frac{1}{\nu}\frac{\partial(\nu < \upsilon_{\mathrm{z}}^2 >)}{\partial z} = -\frac{\partial \Phi}{\partial z}, \tag{1}$$

$$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho, \tag{2}$$

$$\frac{\partial}{\partial z}\left[\frac{1}{\nu}\frac{\partial(\upsilon\nu < \upsilon_{\mathrm{z}}^2 >)}{\partial z}\right] = -4\pi G \rho, \tag{3}$$

[5] In the model stars are distributed along the z axis not too far from the axis of rotation so that their motion cannot be greatly influenced by rotation.

[6] Jeans (1919) pointed out that stellar encounters do not play an important role in the equilibrium of the Galaxy and considered the local neighborhood as a stellar system having a steady configuration governed solely by the gravitational field.

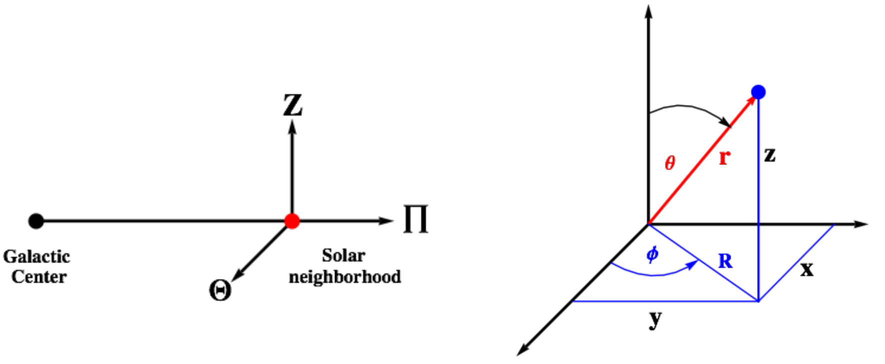

Fig. 3: Coordinate systems.

$$\Sigma(z) \equiv \int_{-z}^{z} \rho(z')dz' = \frac{1}{2\pi G\nu} \frac{\partial(\nu < v_{\mathrm{z}}^2 >)}{\partial z}. \tag{4}$$

Since on the left part of equation (3) we have observables, we can estimate the spatial density ρ. Likewise we can estimate $\Sigma(z)$. The observations, furthermore, give us the Luminosity and luminous mass. The tracers are generally less than a kilo parsec from the galactic plane.

Oort (1932) (Tab. 34) measured a total mass for visible stars of about $0.0378\,M_\odot$ pc^{-3} and a dynamical mass density $0.092\,M_\odot$ pc^{-3}. There is an indication, furthermore, that the invisible matter density is increasing toward the galactic plane, which is fine if we are dealing with dissipative baryonic matter. Oort in his analysis made the strong hypothesis (equilibrium) that stars passing through the galactic plane are well mixed, that is the number moving up from below is similar to that of the stars crossing in the other direction (dynamical equilibrium). That is the number density $\nu(z)$ remains constant in time and we can solve estimating the vertical force K_{z}. Here $K_{\mathrm{z}} = -\frac{\partial\Phi}{\partial\theta} = \frac{1}{\nu}\frac{\partial(\nu<v_{\mathrm{z}}^2>)}{\partial z}$ is what is normally used since in this way we reduce the errors by using only the first derivative of the velocity dispersion (rather than the second derivative as given above) at various distances from the galactic plane and for various types of stars. He found a significant discrepancy between the total density observed and the density that we measure in visible stars. The discrepancy remains even accounting for, see also later work, possible absorption not accounted for, gas and dust. Indeed it was (at the epoch of the study) not yet completely clear how much of the effect may still be due to absorbing dust and which is due to faint stars not yet detected.

In cylindrical coordinates the Poisson equation as derived by Jeans may be written as[7]:

$$K_{\mathrm{R}} = \frac{\partial\Phi}{\partial R} = \frac{1}{\nu}\frac{\partial}{\partial R}(\nu\langle\Pi^2\rangle) + \frac{\langle\Pi^2\rangle - \langle\Theta^2\rangle}{R}, \tag{5}$$

$$K_{\mathrm{z}} = \frac{\partial\Phi}{\partial z} = \frac{1}{\nu}\frac{\partial}{\partial z}(\nu\langle Z^2\rangle), \tag{6}$$

[7]In the presentation these equations are derived from the Boltzmann equation.

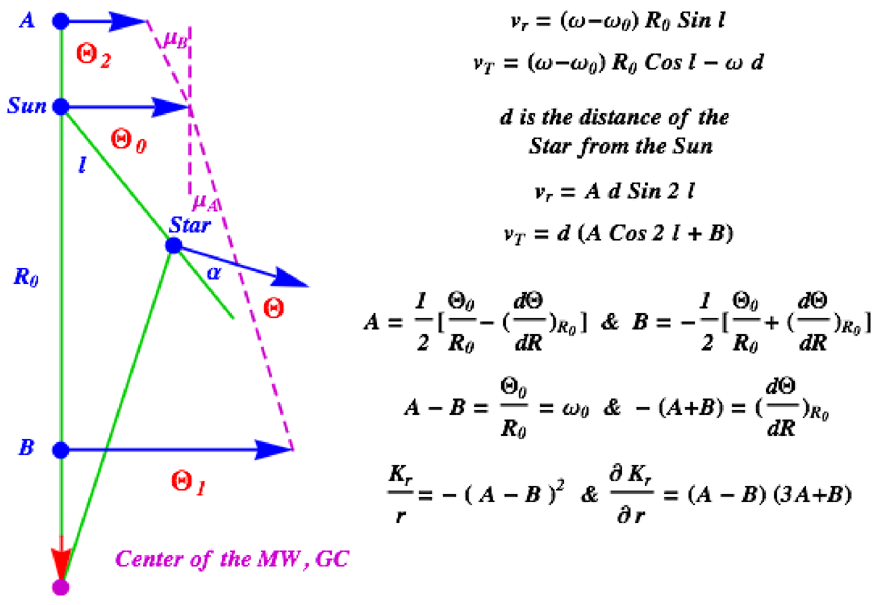

Fig. 4: Differential rotation of the MW and Oort constants.

where Π, Θ, Z are the velocities along the axes R, θ and z. The term $\frac{\partial}{\partial \theta}(\nu \langle \Theta^2 \rangle) = \nu \frac{\partial \Phi}{\partial \theta} = 0.0$ since the field about the z axis is assumed to be symmetric and the derivative of the potential vanishes. These are the perused Jeans equations (Jeans, 1922) where it was assumed $\langle \Pi Z \rangle = 0$ due to the assumption that the velocity distribution in R and Z are identical (see Oort, 1965a, for further details) , an assumption that however may not be true outside of the galactic plane. The Poisson equation $\nabla^2 \Phi = 4\pi G \rho$ for an axially symmetrical system (cylindrical coordinates) becomes

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},$$

and in our case this leads to

$$\frac{\partial K_{\mathrm{R}}}{\partial R} + \frac{K_{\mathrm{R}}}{R} + \frac{\partial K_{\mathrm{z}}}{\partial z} = -4\pi G \rho.$$

In terms of the galactic rotation constants (Oort's constants shown in Fig. 4) A and B we have $\frac{\partial K_{\mathrm{R}}}{\partial R} = (A-B)(3A+B)$ and $\frac{K_{\mathrm{R}}}{R} = -(A-B)^2$. The first two terms are of minor importance and in any case near the Galactic plane they are known rather accurately.

Hill (1960) recomputed the contribution of the visible stars, took into account the contribution by interstellar gas and evaluated the density accounting for the whole Poisson equation where the first two terms give, adopting the value of A and B available at that time, a density of 9.2×10^{-24} g cm^{-3} ($0.135\, M_{\odot}$ pc^{-3}). This compares with a density of visible stars 3.3×10^{-24} g cm^{-3} to which we need to add an Hydrogen (H_2 + 10% H) and Helium density of about 1.6×10^{-24} g cm^{-3} and 0.6×10^{-24} g cm^{-3} for faint Me stars to reach a total visible mass of about $5.5 \times$

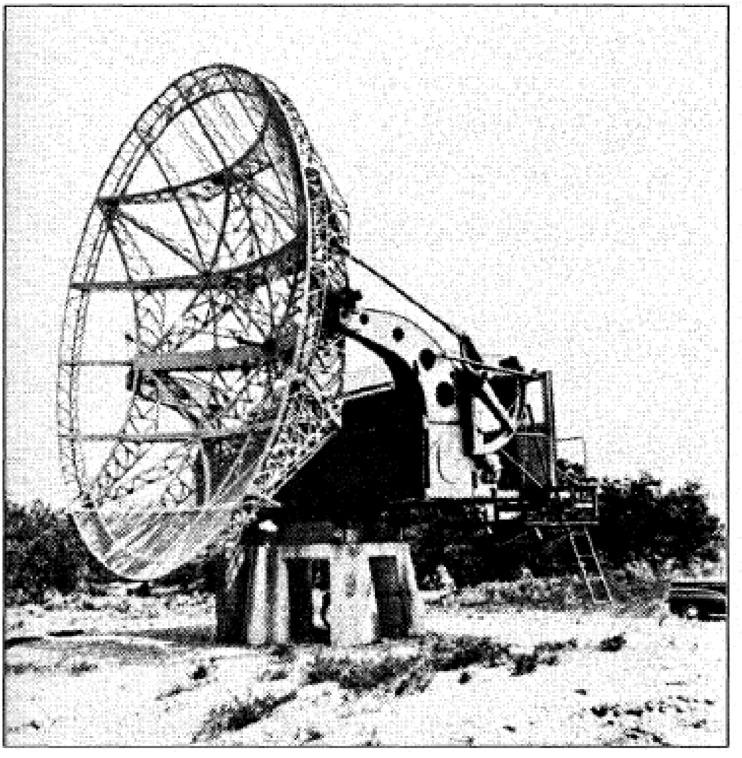

Fig. 5: The Würzburg radar reflector at Kootwjik (from the Van Woerden article, van Woerden & Strom, 2006).

10^{-24}g cm^{-3} ($0.08\,M_\odot$ pc^{-3}). We have a density difference between the dynamical and visible densities of about 3.8×10^{-24} g cm^{-3} ($0.06 M_\odot$pc^{-3}). The estimated values of A and B changed with time, however the value of the first two terms is order of magnitudes smaller then the derivative of K_z to estimate the density on the galactic plane so that it can be, as mentioned above, easily disregarded. The value estimated by Hill (1960) is considerably larger than the value given by Oort (1932), however, in 1960 (Oort, 1960; see his review paper Oort, 1965a), Oort also derives a larger value: $0.15\,M_\odot$ pc^{-3}. For values and details above the galactic plane see the 1960 paper. A newer analysis by Hill et al. (1979) estimates a total mass of known matter in the solar neighborhood of $0.108\,M_\odot$ pc^{-3}, a dynamical mass $\rho_0 = 0.14\,M_\odot$ pc^{-3} so that we still remain with an unexplained mass of $0.03\,M_\odot$ pc^{-3}.

Van de Hulst visited Leiden at the beginning of 1944 and following fundamental intuition and suggestion by Oort[8] discovered that neutral hydrogen should show a line in absorption or emission depending on whether the spin temperature would be large or smaller than the temperature of the background radiation field at these wavelengths (Van de Hulst, 1945).

[8]Oort was also aware of the work by Reber (1940) and sensed that radio astronomy could overcome the strong limitation by which optical astronomy was affected due to the interstellar absorption.

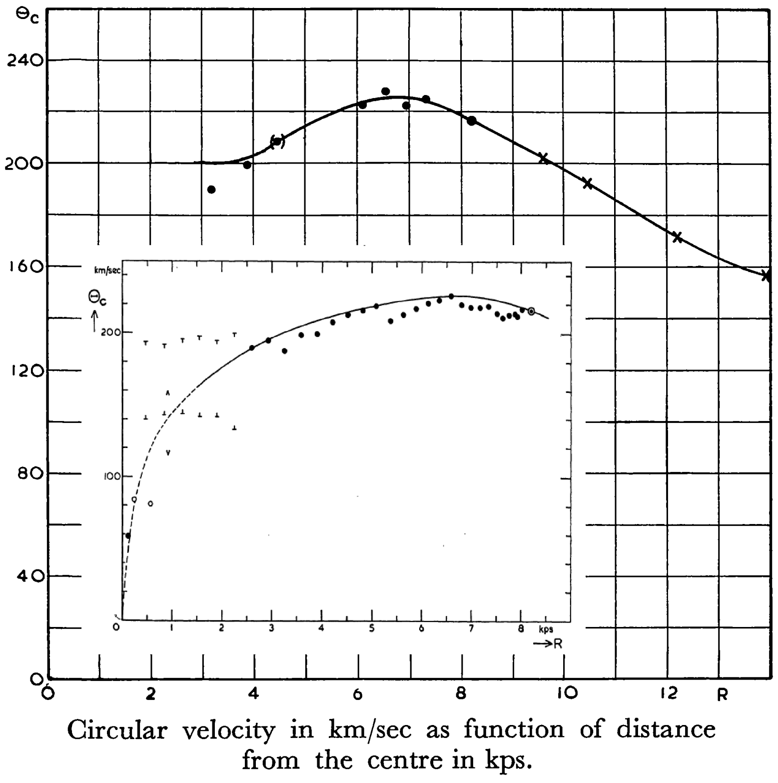

Fig. 6: The first rotation curves of the Milky Way by van de Hulst et al. (1954) and in the inset by Kwee et al. (1954).

The search for the line started[9] both in the US (Ewen & Purcell, 1951) and in The Netherlands (Muller & Oort, 1951), with the use of radar reflector presented in Fig. 5. These observations led to the first rotation curves[10], Kwee et al. (1954) and van de Hulst et al. (1954), Fig. 6.

Toward the end of the seventies Bosma (1978) gives a fundamental contribution on this field showing a sample of galaxies, 25, with flat rotation curve[11]. The time

[9]Van Woerden (van Woerden & Strom, 2006) gives a detailed and fascinating report about the early discovery and how Oort not only was trying to build a radio telescope but how he was capable of accelerating the times by using at Kootwjik a Würzburg radar reflector used by the German forces during the war. The discovery US–NL is also a remarkable example of scientific respect and correctness.

[10]These data were used by Schmidt (1956) in his model of the Milky Way. Kwee et al. noticed that the curve remain about flat at the maximum (see also van de Hulst et al.) and however he assumed (obviously based on the knowledge of the time) in his extrapolation a declining rotation velocity.

[11]With the advent of large National and International facilities the way of making research in many US Institutes and Departments changed drastically and finally the astronomical community at large had access to state of the art facilities. This was a great achievement and the history of the KPNO illustrates the great progress toward a different future thanks also, among others, to great scientists as Aden Meinel and Nicholas Mayall. On the other hand during the transition time, and the sixties and seventies were a transition time, various matters had to be understood in order to

was coming, as mentioned by Hill et al. (1979), to try to investigate which kind of matter we were missing. But, before jumping that far, let's further look into our neighborhood to refine the estimates of the missing mass.

Bahcall & Soneira (1980, 1984) and Bahcall (1984c, 1986) had the bright, and therefore very logical, idea to improve on the analysis using a detailed model of the Galaxy based on the structure observed also in external Galaxies. They also matched the large amount of stellar data (counts) with the model. In other words, improving on the model of Schmidt (Schmidt, 1956; and however see his review paper: Schmidt, 1965), it is assumed that the Milky Way is composed by a disk, a spheroid and if needed a spherical halo. The model is used to calculate the expected results in term of observational parameters (number of stars per unit magnitude and color bin—they use observations in 17 fields) and iterate until the model results agree with the observations. To have a feeling for the quality of the fit we reproduce one of their figures – Fig. 7 (Bahcall & Soneira, 1984).

Bahcall & Soneira (1980) use, in their model, the mass density determined by Oort[12] of $0.15\,M_\odot\,\mathrm{pc}^{-3}$ of which 0.045 is due to visible matter, $0.045\,M_\odot\,\mathrm{pc}^{-3}$ interstellar matter and $0.06\,M_\odot\,\mathrm{pc}^{-3}$ is the missing unidentified mass. Assuming a column of $700-1000$ pc the column density is $\sum_\odot = 75\,M_\odot\,\mathrm{pc}^{-2}$, and we adopt this value to estimate the central surface density in an exponential disk $\sum\limits_0 = \frac{\sum_\odot}{e^{\frac{-r}{R_\mathrm{d}}}}$ with $R_\mathrm{d} = 3.5$ kpc.

The rotation velocity, Fig. 8, of such a disk (Freeman, 1970) is given by the equation 7 where I and K are the Bessel functions and $Y = \frac{r}{2R_\mathrm{d}}$.

$$[4\pi\Sigma_0 G R_\mathrm{d} Y^2\{I(0,Y)K(0,Y) - I(1,Y)K(1,Y)\}]^{\frac{1}{2}}. \tag{7}$$

For the spheroid, see Bahcall & Soneira (1980) and references therein, we derive, with $B = 7.669$ and $r_\mathrm{e} = 1/3$ of the solar distance from the galactic center, the normalization constant C from the total mass, $M_\mathrm{S} = 0.33 \cdot 10^{10} M_\odot$, estimated by Bahcall (1984a).

$$\rho_\mathrm{S}(r) = C\frac{e^{-B(r/r_\mathrm{e})^{7/8}}}{(r/r_\mathrm{e})^{1/4}}. \tag{8}$$

The contribution given by this spheroid to the velocity field is illustrated in Fig. 8. Using solely the disk and spheroid we are unable to reproduce the observed velocity of the Sun as very clearly as pointed out by Bahcall & Soneira (see also Oort's constants).

The idea of a massive halo of unseen matter not only was considered since the work by Oort, but it seems to be needed to avoid the instability of a rotating thin disk. Kalnajs (1972) demonstrated that rotating thin disks are unstable. Ostriker & Peebles (1973) using numerical simulations confirmed the instability of a cold disk and also showed that a massive halo would be necessary for our galaxy, and other spirals, in order to have a stable rotating disk. This was a fundamental step forward for the dynamics of the disk and for the model of the galaxy. The mass and the composition, at the time a possibility was, as hypothesised by Oort, a population of

improve the system and the telescope time allocation system.

[12] Oort (1965a) states that the most probable present value for the matter density in the solar neighborhood is $0.148\,M_\odot\,\mathrm{pc}^{-3}$ (10×10^{-24} g cm^{-3}) of which 40% must be due to stars or gas of unidentified type.

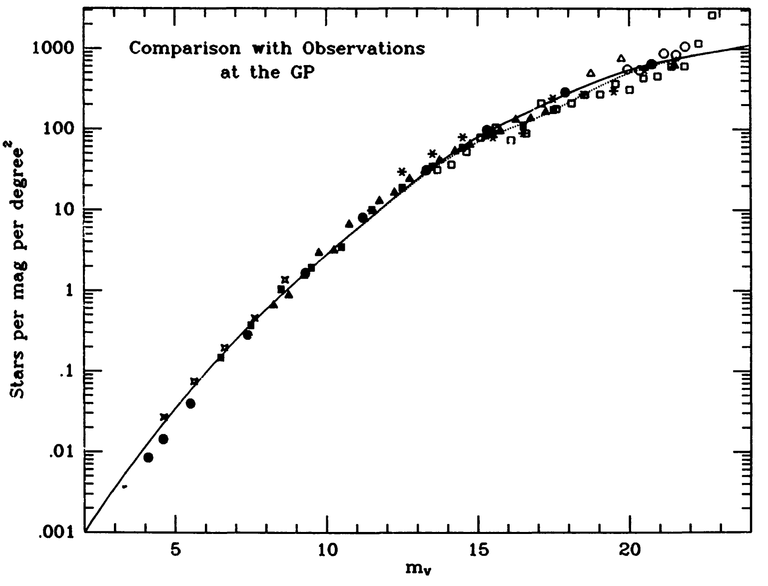

Fig. 7: From Bahcall & Soneira (1984). Differential star counts mag^{-1} deg^{-2} for the galactic pole. Solid curve is predicted by the standard model. Data from Seares et al. (1925) as reduced to the visual band in Bahcall & Soneira (1980) are plotted as filled circles, data from Weistrop (1972) with Faber et al. (1976) corrections as filled squares, data from McLaughlin (1983) as open crosses, data from Reid & Gilmore (1982) as filled triangles, data from King (Chiu, 1980) as asterisks, data from Jarvis & Tyson (1981) as open squares, data from Peterson et al. (1979) as open triangles, data from Kron (1978, 1980) as open circles.

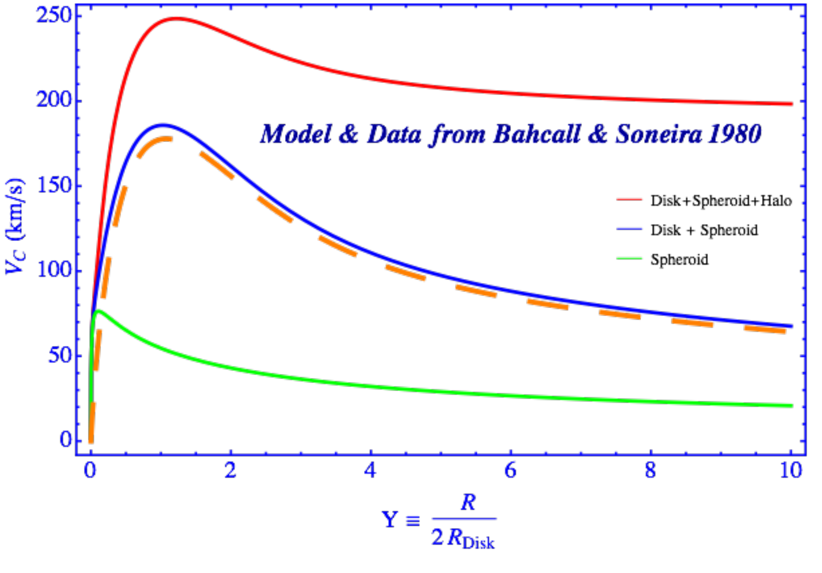

Fig. 8: Rotation curve of the Milky Way.

stars even fainter than K, M stars, is still unknown. We will have to wait till the end of the seventies for a better understanding of the primordial nucleosynthesis and the modelling of an halo of non baryonic dark matter. With a density distribution

$$\rho_{\mathrm{H}}(r) = \frac{\rho_{\mathrm{H}}(0)}{1 + \left(\frac{r}{R_{\mathrm{core}}}\right)^2}, \tag{9}$$

we constrain the unknown parameters by assuming that the rotational velocity observed near the Sun, or more precisely at large distances from the rotation axis, will coincide with $V_{\mathrm{C}}(\infty) = \lim_{r\to\infty} \sqrt{\frac{2G\int_0^r \rho_{\mathrm{H}}(r) 4\pi r^2 dr}{r}} = 43.43 \times 10^{-5}\sqrt{\rho_{\mathrm{H}}(0) R_{\mathrm{d}}^2}$ and $\rho_{\mathrm{H}}(0)$ is derived from the observed missing mass density in the solar neighborhood: $0.06\,M_\odot\,\mathrm{pc}^{-3}$ according to Oort. I computed the curve of Fig. 8 using these parameters somewhat adjusted. Note that I did not try to optimize the fitting with the observations since, as we will see, things changed somewhat with the most recent work and I developed the above solely for illustrative purposes (for a detailed model see, among others, McGaugh (2016). According to Bahcall (1985) the density of the unseen matter is somewhat concentrated on the galactic plane[13] (not supported however by later observations and analyses). In 1984 John Bahcall in a series of papers (Bahcall, 1984a,b,c) concludes that the missing mass is of about $0.1\,M_\odot\,\mathrm{pc}^{-3}$ *or* $30\,M_\odot\,\mathrm{pc}^{-2}$ (unobserved disk material 50% of the total). For the model and parameter values see Bahcall (1985).

The next significant effort is carried out by Kuijken & Gilmore (1989a,b,c); Kuijken (1991). They obtained the very important result that there is no missing mass on the galactic disc near the Sun though there is certainly what is due to a massive dark Halo. The result is fundamental because in case of non dissipative baryonic matter there is no need to join in a plane the baryonic visible matter (play with $\frac{\delta\rho_{\mathrm{baryonic}}}{\rho_{\mathrm{DM}}}$). These authors, avoiding some of the internal inconsistencies they detected in previous data, measure a density of matter $\rho_0 = 0.10\,M_\odot\,\mathrm{pc}^{-3}$ and a surface density $\frac{|K_{\mathrm{z}}(1.1\mathrm{kpc})|}{2\pi G} = 71.6\,M_\odot\,\mathrm{pc}^{-2}$ and $48 \pm 8\,M_\odot\,\mathrm{pc}^{-2}$ for the disk. Refinement of these data follow in time and, as we see from Tab. 2, are based over a very large amount of recent observations. The small amount of DM we can measure locally is part of the large Halo that flattens the rotation curve.

The rather recent work by Zhang et al. (Zhang et al., 2013, and references therein) find, as did Konrad Kuijken and Gerard Gilmore, that there is not a significant amount of DM in the disk and estimate a value of $\mathrm{DM}_{\mathrm{local}} = 0.0075 \pm 0.0021\,M_\odot\,\mathrm{pc}^{-3}$. This is in good agreement with what is expected from the density of the Halo DM in the solar neighborhood (this can be easily found out by using, for instance, an isothermal distribution for the halo).

The rotation curve of the Milky way has been recently extended at large distances from the galactic center. It is interesting to note, see the Fig. 9 obtained with the data from Bhattacharjee et al. (2014) and Huang et al. (2016), that the rotation velocity decreases steadily in the outermost regions. According to Huang et al. (2016) the curve remains flat at about 240 $\mathrm{km\,s^{-1}}$ up to a distance of 25 kpc from the galactic center to decrease steadily afterwards, Bhattacharjee et al. (2014) indeed find a

[13]The student may develop the concept of concentration in the galactic plane in relation to the characteristics of various DM particles and baryonic matter.

Unit $M_\odot \mathrm{pc}^{-3}$	Tot. Density	Lum. Density	DM Density
Kapteyn	0.099		
Oort 1932	0.092	0.038	
Hill 1960	0.135	0.08	0.06
Oort 1960 - 1965	0.15		
Hill - Hilditch 1979	0.14	0.11	0.03
Bahcall 1984	0.185		~ 0.09
Kuijken & Gilmore	0.10		0.10
Holmberg & Flynn	0.102	0.095	~ 0
McGaugh			0.009
Zhang	$\sum = 67$	$\sum = 42$	0.0075

Tab. 2: The density of matter measured by Kuijken & Gilmore (1989a,b,c); Kuijken (1991).

steady decrease up to about 200 kpc. The mass of the DM Halo estimated by Huang et al. (2016) is $M_{\mathrm{MW}} = 0.90^{+0.07}_{-0.08} \times 10^{12}\, M_\odot$ (in agreement with the value $M_{\mathrm{MW}} \succ 6.8 \pm 4.1 \times 10^{11}$ estimated by Bhattacharjee et al. (2014)). The Huang model of the Halo gives a local DM density $\rho_{\mathrm{DM,Local}} \approx 0.32 \pm 0.02$ GeV cm^{-3} ($0.084\, M_\odot$ pc^{-3}).

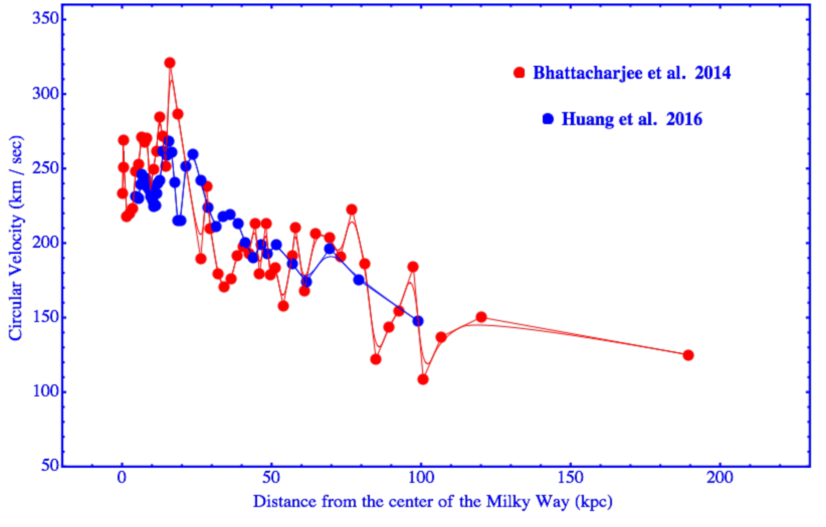

Fig. 9: Recent observations on the velocity curve of the Milky Way.

These estimate of the mass of the Milky Way and related DM are now rather robust, see also Rashkov et al. (2013) for a critical discussion. Finally for a very detailed analysis of the rotation curve and a derivation of the Oort's constant see McGaugh (2016). This authors estimate $\rho_{\mathrm{DM,Local}} = 0.009\, M_\odot$ pc^{-3} (0.34 GeV cm^{-3}) in very good agreement with the latest estimates.

We discussed the local density of matter that led us to touch upon the model of the Galaxy. In order to explain the flat rotation curve in the Milky Way we need a large Halo dominated by DM. The analysis and the models for external galaxies follows what we discussed for the Galaxy, spiral galaxies show flat rotation curves at large distances from the center.

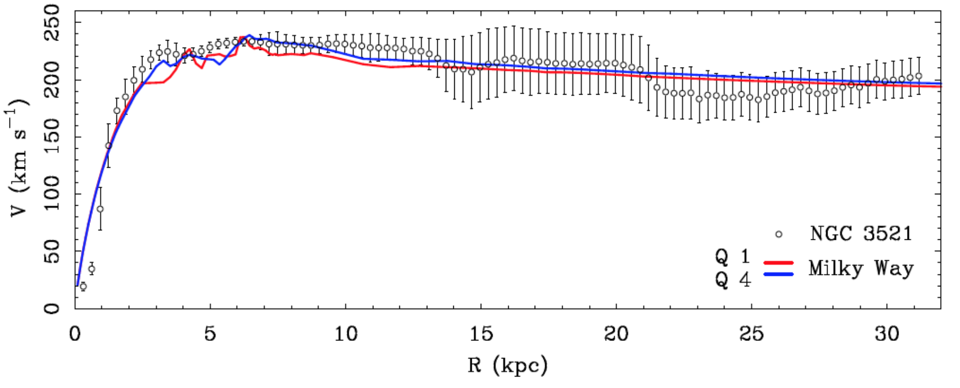

Fig. 10: From McGaugh (2016). The rotation curve of NGC 3521 is from de Blok et al. (2008).

In the late fifties the Burbidges (Margaret and Geoffrey) were deeply involved in the study of external galaxies and toward the end of that decade started to publish, in collaboration with K.H. Prendergast, a series of fundamental papers on the dynamics of galaxies. Margaret Burbidge an excellent observer, Geoffrey Burbidge an innovative and genial theoretician and Kevin H. Prendergast the expert in dynamics. When I started to work with Merle F. Walker (Lick Observatory, Mt Hamilton – 1964) their papers were almost like a bible. At that time the spectrographs allowed low dispersion and spatial scale with photographic plates (in fact on the fast cameras astronomers used film) that were not that much sensitive after all. In a few words the observations and therefore the modeling would not allow observations of the faintest outskirt of the galaxies. During the seventies we gained awareness of the flat rotation curves and the need of the halo starting with the work of Robert and Rots (Roberts & Rots, 1973) who showed with Radio observations that not only M31 had a flat curve, see the observations by Rubin & Ford (1970), but also M81 and M101. Later an other milestone in the field, in addition to the continuous and dedicate work by Vera Rubin and Kent Ford, has been the paper by van Albada, Bahcall, Begelman and Sancisi (van Albada et al., 1985). In this galaxy, NGC 3198, the flat rotation curve extend to 30 kpc, almost four times the distance of the Sun from center of the Milky Way and has DM density (at a distant of 8 kpc from the center) $\rho_{\rm halo}(8\ {\rm kpc}) = 0.0042\, M_\odot\, {\rm pc}^{-3}$ that is comparable to that measured by Zhang (Zhang et al., 2013) in the solar neighborhood $\rho_{\rm halo}(\odot) = 0.0065\, M_\odot\, {\rm pc}^{-3}$. The minimum amount of dark matter associated with NGC 3198 inside 50 kpc is probably at least 6 times larger than the amount of visible matter. In Fig. 10 we show, from the paper by McGaugh (2016), the similarity of the rotation of the Milky Way with that of an external galaxy (NGC 3521).

Nowadays we have a large amount of data on rotation curves and a rather interesting compilation of rotation curves has been done by Sofue (2016). As can be seen from the Fig. 11 (courtesy of Prof. Sofue) various galaxies show rotation curves that are flat and in some cases smoothly declining at very large distances. This gave early evidence that the masses of galaxies are larger than we thought and Ostriker Pee-

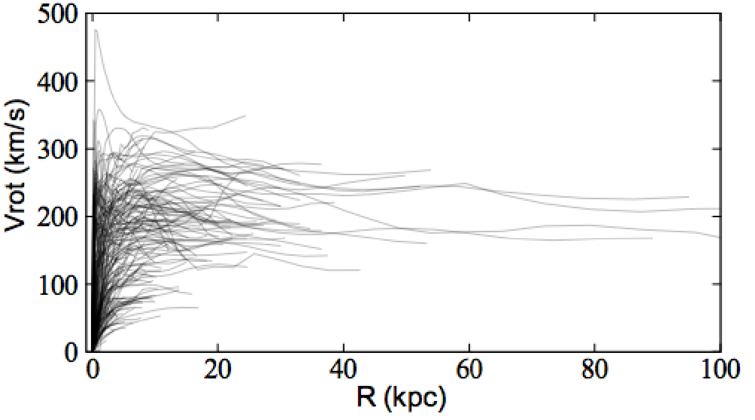

Fig. 11: An ensemble of rotation curves of galaxies, courtesy of Sofue (2016).

bles and Yahil (Ostriker et al., 1974), as did Zwicky in 1933 (Zwicky, 1933), use the new mass estimates of giant spiral galaxies to estimate a mean mass density of the Universe $\sim 2 \times 10^{-30}$ g cm^{-3} corresponding to $\Omega \equiv \frac{\rho}{\rho_{\rm crit}} \simeq 0.2$. Joanna Jalocha and her team made some rather interesting remarks during her lecture at the first Kielce Cosmology School and presented a very interesting way of rotation curves analysis, see in particular Jałocha et al. (2008). I refer to that work for the important analysis they developed. Here I would like only to call attention to the observations of NGC 4736 and refer to their analysis also discussing the recent observations by Lang et al. (2017).

Chincarini & Walker (1967) observed the rotation curve of the central region of NGC 4736 and Bosma et al. (1977) measured the rotation up to a distance of 10.5 kpc from the center. The argument made by Jałocha et al. (2008) essentially states that if the decay in velocity is faster than $\frac{1}{\sqrt{r}}$ then we have a problem. In the case of NGC 4736 the argument has been based on the last point observed by Bosma et al. (1977). The data by Bosma el al., and relative errors, are plotted in red in Fig. 12 while the $\frac{1}{\sqrt{r}}$ law is represented by the dashed line (normalized at the point at 4 arcmin).

The deviation of the last point is rather small and almost within error, indeed the error of the last observation at 6 arcmin must be larger than what has been quoted.

I measured the values I plotted from the Fig. 10 in Bosma et al. (1977) and I simply measured a few points from the observations plotted in Fig. 19 of Frank et al. (2016) dividing by $(\sin i)$ where i is the inclination of the galaxy given in that paper. It seems that the error in the last point of Bosma et al. has a larger error than estimated, may be systematic.

Obviously a more accurate work could be done and should eventually be done, here I only wanted to give a warning. On the other hand recently Genzel et al. (2017), Lang et al. (2017) find that early falloff of high z galaxies my be a common feature and the dominant mass component is baryonic matter. The rotation velocity

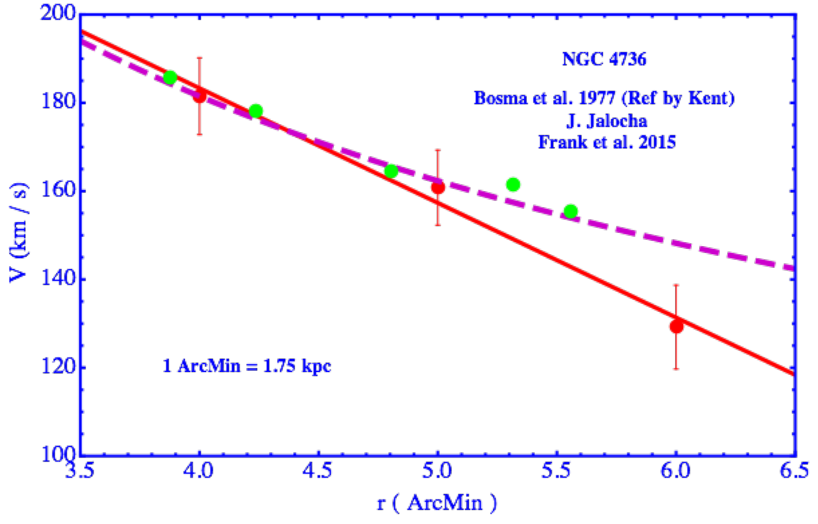

Fig. 12: The decaying velocity curve of NGC 4736. In red the observations by Bosma et al. (1977) fitted by the continuous line. The dashed curve the $\frac{1}{\sqrt{r}}$ law and the green points from the work by Frank et al. (2016).

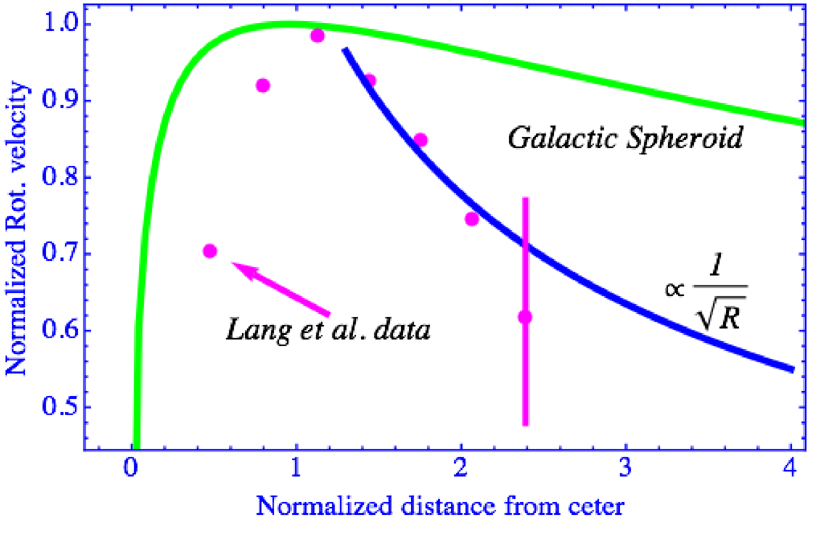

Fig. 13: The data by Lang et al. (2017) (red points) have been measured on their figure. The green and blue continuous line are given as reference of the rotation curve due to a spheroid and decay proportional to $\frac{1}{\sqrt{R}}$.

falls off is about proportional to $\sqrt{r}$, Fig. 13. The sample galaxies are at a distance $0.6 < z < 2.6$, isolated with mass in the range $10.6 < \log_{10} M_*/M_\odot < 11.1$ and a rather large $4 < R_{1/2} < 9$ kpc. Compare to local galaxies the authors find $\left(\frac{V_{\rm rot}}{\sigma_0}\right)_{\text{high } z} \sim 4-9 < \left(\frac{V_{\rm rot}}{\sigma_0}\right)_{\rm local} \sim 10-20$.

The authors explain all this with a concentration of baryonic matter that is larger than the concentration of DM, that is we have baryons dominated galaxies with a pressure term (velocity dispersion) rather large. These may be galaxies in the process of forming with matter that is collapsing. The question is how will these galaxies transform in what we observe at $z \sim 0$ and how is working the 'accretion carrying, or developing afterwords, angular momentum.

In conclusion the estimate of the mass, both dynamical and visual, for the environment of the solar system and for the MW is very robust. Likewise the estimates of the mass via the rotational velocity curve of external galaxies are in excellent agreement with the mass derived, see lecture II, via weak lensing and the rotation curves remains, in many cases, flat up to 30–40 Kpc to slowly decrease at greater distance from the center. In the case of MW & M31 the rotation curve seems to extend up to few hundred Kpc in spite of a separation between the two galaxies of only 600 Kpc. We need further information on how a galaxy gains angular momentum and preserves it in spite of strong gravitational perturbations that exist in some cases. Finally the work of Genzel et al. (2017), Lang et al. (2017) opens a new window in the evolution and formation of galaxies.

A few personal remarks may help understanding the way I witnessed and understood the developments about 50 years ago. I noticed in fact that quite often personal notes, especially those of an ordinary scientist, move the curiosity of young scientists stimulating them to satisfy their curiosity in research.

In the early seventies the astronomical community at large became aware of a missing mass problem[14]. The book *Morphological Astronomy* written by Zwicky fascinated some of us and, of course, the discussions I had with Zwicky were illuminating and creative. Rood and I were engaged in observing as many redshifts as possible to better understand the missing mass in clusters since we (Rood was working on this problem in collaboration with Thornton Page since a few years) were strongly convinced of the reality of the effect. For a few years H. Rood had been chasing accurate estimates of the missing mass in clusters and discussing it with colleagues. The instrumentation was also in a fundamental transition phase with the advent of the image tubes and when I went to the Johnson Space Flight Center (1969) Thornton Page, after a couple of months I was there, asked me to take his place[15] in the collaboration with Rood since I was an expert in image tubes. That is the way my work on clusters (leading later to LSS) started.

The faint outskirt of galaxies far from the center of rotation was too weak to

[14]In the early sixties most astronomers would not believe Zwicky's results and the existence of DM. Burbidge and Burbidge in their Hercules paper (Burbidge & Burbidge, 1959) state:

> "Suffice it to remark here that the values obtained by Zwicky are often unrealistic, since he has never accepted any of the revisions in the distance scale since Hubble's (1936) value of the red-shift constant of 526 km/s per 10^6 pc."

[15]Thornton was very busy at the time. His dream was to have a small telescope on the surface of the Moon and carried there by one of the Apollo flight. Unfortunately (or fortunately) the HST project was on the horizon.

be detected and measured even using the fast camera nebular spectrographs and the most sensitive photographic film. In the classical series of papers by Burbidge, Burbidge and Prendergast (Burbidge et al., 1959, and following years) on the rotation curve and mass distribution of galaxies, data at large distances from the center were lacking[16].

Radio 21 cm observations in the seventies were becoming fundamental to the estimate not only of the Hydrogen content but also to the measure of the velocity field in spirals. Fundamental had been, for instance, the dissertation by Bosma (1978) on rotation curves of external galaxies observed in 21 cm. With the advent of the RCA image tube Rubin and Ford would start, after extending the work of Babcock on M31 (Babcock, 1939), a series of fundamental observations to measure the rotation of galaxies also at large distances from the center (see Rubin & Ford, 1970, and subsequent papers).

In the sixties observational work could be done also using also National Facilities[17] since the Kitt Peak National Observatory in Arizona as announced by N. U. Mayall in 1961 ("... The 84-inch stellar telescope will be made available for use by visiting astronomers starting September 15, 1964 ...") started to operate. The paper by Rood, Page, Kintner and King (Rood et al., 1972) on the Coma cluster contributed strongly in convincing the community about the missing mass problem, it was clear however that many astronomers were still skeptical about it. Actually the publication year is some misleading since part of the results were known sometime earlier due to Herb's dissertation and the fact it took a long time to publish that work[18] as Herb Rood told me, on the other hand the program he started with Page on clusters of galaxies was proceeding well. The paper by Ostriker et al. (1974) is quite significant since it is clearly demonstrated and discussed that the mass to light ratio of galaxies is very large, about 200 $(M/L)_\odot$. That the matter was somewhat still confused it is clear from the suggestion by Einasto et al. (1974)[19] that the large

[16]The research carried out starting in the years 1959–1960 is very significant in various aspects. The study of the rotation curve of NGC 5128 uses essentially the Kepler's III law and the later paper on NGC 1068 is in collaboration with Prendergast. The paper on the Hercules cluster *(q.v.)* uses the virial theorem to estimate the cluster mass. However it is significant for the comment regarding Zwicky's results referred to above which practically denies the existence of DM and suggest the expansion of clusters. It is a fact, that Zwicky did not accept all the revisions of H_0, however this is not the point. (See also the Berkeley symposium; Neyman 1961.) [*particularly Page (1961) and Lovasich et al. (1961); see also the 1961,* Conference on the Instability of Systems of Galaxies*, Santa Barbara; Neyman et al. 1961 – Eds.*]

[17]National facilities, as I said, helped a lot. On the other hand during the early times, sixties & seventies, the observer had some disadvantages respect to scientists who used paper, pencils and available literature data. Whoever had an idea worth of observational development had to make a plan, apply for resources, apply for telescope time making a science case detailing his/her plan. Tom Kinman once asked me to present the work on LSS (middle seventies) as KPNO high light to NSF. I refused upset because the TAC (early seventies) had flanked my latest request for telescope time (the pioneering work on the LSS structures). Stupid of me, but give an idea of the tension (at least on my part). The observer finally after getting the time needed to reduce the data etc. etc.The time gap between ideas, plans and publication was rather long. The system is now rather robust and efficient with ground based and space facilities & rather unbiased committees. The young scientist should be aware and deeply convinced that getting original and new data supporting ideas and new models is, no matter what, very exciting and that a good and careful work always pay off.

[18]I was told that Ivan King was very meticulous in checking carefully all the details and then they had the technical times needed to the referee and editing procedures.

[19]Here they use for pairs the data by Page (1970) and by Karachentsev (1966), However Tab. 1 (parameters of coronas) and Tab/ 2 (parameters on pairs) are missing (?) from the Nature paper by EKS.

M/L ratio in galaxies may explain the large mass discrepancy observed in clusters. The study of the Perseus galaxy cluster by Chincarini & Rood (1971)[20] evidenced a very large velocity dispersion that coupled to the irregular shape of the cluster did not exclude, according to them, an expansion component of the Perseus cluster of galaxies, see however Kent & Sargent (1983).

By the end of the seventies the fact that we had a missing mass problem, or missing light as David Schramm liked to refer to it, on galaxies and on larger scales, clusters in particular, was widely accepted.The hunt for identifying DM also had started and for a few years there had been a lot of discussions about massive neutrinos. In this period I was planning the IAU meeting in Crete (1983) in collaboration with George Abell and George Contopoulos. I recall asking Martin Rees for some advice. Witnessing the great interest in this topics by the astronomical community, I planned to possibly have a strong section on neutrinos. He told me not to worry too much, the fashion will pass he said and ... it did. The possibility of non-baryonic matter "... inos" was discussed in this period especially in various schools of Cosmology. And, as we will see, quite quickly Dark Matter was identified with non-baryonic dark matter and the search for exotic particle was also on its way.

It is always healthy to creatively doubt since this is a big push toward searching the truth.

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[20]Part of the observing program on the mass estimate of clusters of galaxies started by Page and Rood at KPNO.

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