Non-integrability of the n body problem

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学术报告

Title : Relativistic and quantum dynamics of a charged particle stored in a Penning trap

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Maria Przybylska received the PhD degree and Habilitation in Theoretical Physics from Nicholaus Copernicus University in Torun, Poland in 1999 and 2009, respectively. Since 2010 she has been working in Institute of Physics at University of Zielona Gora. Her research activity concerns effective approaches to search of integrable dynamical systems based on the Kovalevska, Ziglin and Morales–Ramis methods, analysis of the dynamics for certain quantum systems, analytical methods of spectra determination for quantum optics systems in the Bargmann-Fock representation, and relativistic dynamics of charged particles in a Penning trap.

W mechanice nieba, jak i w matematyce u dzikich — trzy znaczy dużo.

Émile Borel

How to solve?

With nice formula ...

$$ax^2 + bx + c = 0.$$

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The n-body problem. Equations of motion.

Newton's equations

$$m_i \ddot{\boldsymbol{r}}_i = -\sum_{j=1}^{n}{}' \frac{m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} (\boldsymbol{r}_i - \boldsymbol{r}_j), \qquad i = 1, \ldots, n$$

A vector $\boldsymbol{r} = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n) \in \left(\mathbb{R}^2\right)^n$ will be called a configuration.
More compact form

$$\boldsymbol{M} \ddot{\boldsymbol{r}} = -\nabla V(\boldsymbol{r}), \qquad V(\boldsymbol{r}) = -\sum_{1 \le i < j \le n} \frac{m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|},$$

where

$$\boldsymbol{M} = \begin{bmatrix} m_1 \boldsymbol{I} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & m_2 \boldsymbol{I} & \ldots & \mathbf{0} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{0} & \mathbf{0} & \ldots & m_n \boldsymbol{I} \end{bmatrix},$$

Hamiltonian

$$H = \frac{1}{2}\boldsymbol{p}^T\boldsymbol{M}^{-1}\boldsymbol{p} + V(\boldsymbol{r})$$

where $\boldsymbol{p} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) \in \left(\mathbb{R}^2\right)^n$; $\boldsymbol{p}_i = m_i\dot{\boldsymbol{r}}_i \in \mathbb{R}^2$. First integrals

$$\boldsymbol{P} = \sum_{i=1}^{n}\boldsymbol{p}_i, \qquad C = \sum_{i=1}^{n}\boldsymbol{r}_i^T\boldsymbol{J}^T\boldsymbol{p}_i, \qquad \boldsymbol{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

These first integrals do not pairwise commute

$$\{C, P_1\} = P_2 \quad \text{and} \quad \{C, P_2\} = -P_1.$$

$2n$ degrees of freedom, $2n$ independent and commuting first integrals are needed for the integrability in the Liouville sense.

Integrability. History.

Heinrich Bruns (1848 – 1919)
In the Newtonian three-body problem in the space, every first integral which is algebraic with respect to positions, linear momenta and time, is an algebraic function of the classical first integrals: the energy, the three components of angular momentum and the six integrals that come from the uniform linear motion of the center of mass.
Über die Integrale des Vielkörperproblems, Lepizig 1887

Corrections I

Poincaré, H.: 1896, C. R. Acad. Sci. Paris 123, 1224:
'...le re´sultat de M. Bruns se trouve donc confirmé ; je suis heureux d'avoir pu complé ter son élégante analyse sur un point de détail'.

H. Poincaré (1854–1912)

Correction II

Paul Painlevé (1863 – 1933)

- 1917 prime minister (6 weeks),
- 1925 premier minister (6 months),
- Painlevé, P.: 1898, ‘Meémoire sur les intégrales premiéres du probléme des n corps’, Acta Math. Bull. Astr. T 15.
- about this work:
  (his) result is more general than Bruns’ one because it only needs algebraicity in linear momenta; yet there are many errors in its proof;

Correction III

Andrew Russell Forsyth
(1858 – 1942)

- Trinity College, Cambridge, graduating senior wrangler in 1881;
- Cambridge lecturer in 1884, and became Sadleirian Professor of Pure Mathematics in 1895,
- He was forced to resign his chair in 1910 as a result of a scandal caused by his affair with Marion Amelia Boys, née Pollock, the wife of physicist C. V. Boys.
- Theory of Differential Equations 6 Volume Set, Cambridge University Press; 2769 pages.
- about his proof (in vol. 3):
  …it is not clear at al!

Correction IV

Sir Edmund Taylor Whittaker FRS, FRSE (1873–1956)
A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Cambridge University Press.

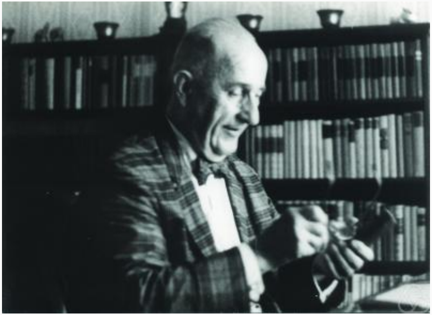
Correction V

William Duncan MacMillan (1871 – 1948)
"On Poincaré's correction to Bruns' theorem". Bulletin of the American Mathematical Society. 1913, 19 (7): 349–355.

Correction VI

Hagihara Yūsuke, (1897–1979)
Celestial mechanics : Dynamical principles and transformation theory (vol. 1). Cambridge: MIT Press.

Correction VII



Carl Ludwig Siegel (1896 –- 1981)
Siegel, C. L.: 1936, ‘Uber die algebraischen Integrale des restringierten Dreikrperproblems’, Trans. Am. Math. Soc. 39, 225.

Correction. The end????

BRUNS' THEOREM: THE PROOF AND SOME GENERALIZATIONS
EMMANUELLE JULLIARD-TOSEL
Celestial Mechanics and Dynamical Astronomy 76: 241–281, 2000.
From abstract:
The whole proof is much more rigorous than the previous versions (Bruns, Painlevé , Forsyth, Whittaker and Hagihara).

What we want to prove?

In the mass centre reference frame we fix the energy $h = H(\boldsymbol{r}, \boldsymbol{p}$ and the angular momentum $c = C(\boldsymbol{r}, \boldsymbol{p})$, and we restrict the system to the common level $\mathcal{M}_{h,c}$ of these integrals.

Theorem

If $(h, c) \neq (0, 0)$, then the planar n body problem, with $n > 2$, and positive masses $m_1, \ldots, m_n$, restricted to the level $\mathcal{M}_{h,c}$, is not integrable.

How to prove non-integrability?

Consider system

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} = (x_1, \dots x_n)^T \in \mathbb{C}^n,$$

and assume that we know non-equilibrium particular solution $\boldsymbol{\varphi}(t)$ the substitution $\boldsymbol{x} = \boldsymbol{\varphi}(t) + \boldsymbol{\xi}$ is applied.

Variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi} = A(t)\boldsymbol{\xi}, \qquad A(t) = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}(\boldsymbol{\varphi}(t)).$$

This is not enough!

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{C}^n, \quad t \in \mathbb{C},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Y} = \boldsymbol{A}(t)\boldsymbol{Y}, \qquad \boldsymbol{A}(t) = \frac{\partial \boldsymbol{v}}{\partial x}(\boldsymbol{\varphi}(t)), \qquad Y \in \mathbb{C}^n.$$

Fingerprints

If the system possesses $k \geq 1$ functionally independent meromorphic first integrals $F_1, \ldots, F_k$, then VEs have k functionally independent rational first integrals.

Ziglin Theory Idea

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = \boldsymbol{A}(t)\boldsymbol{y}, \qquad t \in \mathbb{C}.$$

Fundamental matrix

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Y} = \boldsymbol{A}(t)\boldsymbol{Y}, \qquad \det \boldsymbol{Y}(t_0) \neq 0,$$

Continuation along a loop σ:

$$\boldsymbol{Y}(t) \longrightarrow \widehat{\boldsymbol{Y}}(t) = \boldsymbol{Y}(t)\boldsymbol{M}_\sigma$$

The monodromy group $\mathcal{M} \subset \mathrm{GL}(n, \mathbb{C})$.
A first integral is constant along an arbitrary continuation:

$$f(t, \boldsymbol{M}_\sigma \boldsymbol{y}) = f(t, \boldsymbol{y}) \quad \text{for all} \quad \boldsymbol{M}_\sigma \in \mathcal{M}.$$

Thus: for an integrable system $\mathcal{M}$ cannot be too big!

Equations, equations, equations ...

$$ax + b = 0, \quad ax^2 + bx + c = 0, \quad ax^3 + bx^2 + cx + d = 0, \ldots$$

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0. \tag{P}$$

E. Galois theory! Galois group, etc

Nice fact.

Equation (P) is solvable $\iff$ its Galois group is solvable.

Equations, equations, equations ...

$$a(t)\dot{x} + b(t) = 0, \quad a(t)\ddot{x} + b(t)\dot{x} + c(t)x = 0, \ldots$$

$$a_n(t)x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0. \tag{D}$$

Differential Galois theory! Differential Galois group, etc

Nice fact.

Equation (D) is solvable $\Longleftrightarrow$ its Galois group is solvable.

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{C}^n, \quad t \in \mathbb{C}, \tag{DS}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Y} = \boldsymbol{A}(t)\boldsymbol{Y}, \qquad \boldsymbol{A}(t) = \frac{\partial \boldsymbol{v}}{\partial x}(\boldsymbol{\varphi}(t)), \qquad Y \in \mathbb{C}^n. \tag{VE}$$

Nice implication

If the system (DS) is integrable then the differential Galois group of variational equations (VE) is solvable.

What we need?

We have $n+2$ parameters:

$$m_1, \ldots, m_n, h, c.$$

1. For arbitrary values of these parameters we have to find a particular solution of n body problem.
2. For each particular solution we have to determine the differential Galois group of corresponding variational equations.

Particular solutions of n body problem?

A vector $\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n) \in \left(\mathbb{R}^2\right)^n$ is a central configuration iff

$$\sum_{j=1}^{n}{}' \frac{m_i m_j}{|\boldsymbol{s}_i - \boldsymbol{s}_j|^3} (\boldsymbol{s}_i - \boldsymbol{s}_j) = \mu m_i \boldsymbol{s}_i, \qquad i = 1, \ldots, n,$$

or

$$\nabla V(\boldsymbol{s}) = \mu \boldsymbol{M} \boldsymbol{s},$$

for a certain $\mu \in \mathbb{R}$. It is easy to show that

$$\mu = \mu(\boldsymbol{s}) = -\frac{V(\boldsymbol{s})}{I(\boldsymbol{s})}, \qquad I(\boldsymbol{r}) = \sum_{i=1}^{n} m_i |\boldsymbol{r}_i|^2$$

Animations

Problems:

1. arbitrary n,
2. arbitrary masses,
3. exact values of $\boldsymbol{s}_i$, $i = 1, \ldots, n$.

No hope!

Homographic solutions

For $\boldsymbol{A} \in \mathrm{SO}(2, \mathbb{R})$, $\widehat{\boldsymbol{A}}\boldsymbol{r} = (\boldsymbol{A}\boldsymbol{r}_1, \ldots, \boldsymbol{A}\boldsymbol{r}_n)$.

Solutions

Let $\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$ be a central configuration. Then equations (6) admit solution

$$\boldsymbol{r}(t) = \rho(t)\widehat{\boldsymbol{A}}(\nu(t))\boldsymbol{s}$$

where $\rho(t)$ and $\nu(t)$ are a solution of planar Kepler problem

$$\ddot{\rho} - \rho\dot{\nu}^2 = -\frac{\mu}{\rho^2},$$
$$\rho\ddot{\nu} + 2\dot{\rho}\dot{\nu} = 0.$$

Variational equations

The true anamaly is the independent variable

$$f(\nu)\left(\boldsymbol{X}'' + 2\widehat{\boldsymbol{J}}\boldsymbol{X}'\right) = \widehat{\boldsymbol{G}}\boldsymbol{X},$$

where

$$f(\nu) = \frac{c^2}{\mu\rho(\nu)}, \qquad \widehat{\boldsymbol{G}} = \boldsymbol{I}_{2n} - \frac{1}{\mu}\boldsymbol{M}^{-1}\boldsymbol{H}(\boldsymbol{s}), \qquad \widehat{\boldsymbol{J}} = \operatorname{diag}(\boldsymbol{J}, \dots, \boldsymbol{J}),$$

and

$$\rho = \frac{c^2/\mu}{1 + e\cos\nu}, \qquad \boldsymbol{H}(\boldsymbol{s}) = \nabla^2 V(\boldsymbol{s}).$$

What to do with arbitrary $n > 2$?

F. R. Moulton proved that for every ordering of n positive masses, there exists a unique collinear central configuration of n-bodies. We fix such a central configuration and assume that $\boldsymbol{s}_i = (x_i, 0)$ for $i = 1, \ldots, n$.

First big step: transformation of VE into direct product of equations

$$f(\nu)\left(\boldsymbol{X}_k'' + 2\boldsymbol{J}\boldsymbol{X}_k'\right) = \boldsymbol{G}_k\boldsymbol{X}_k, \qquad \boldsymbol{G}_k = \operatorname{diag}(3 + 2\delta_k, -\delta_k), \quad (1)$$

where $\delta_k = \lambda_k - 1$,and λ_k are eigenvalues of $\boldsymbol{C} = [C_{ij}]$. For $i \neq j$

$$C_{ij} = -\frac{m_j}{\mu|x_i - x_j|^3}, \quad (2)$$

and

$$C_{ii} = -\sum_{j=1}^{n}{}' C_{ij}. \quad (3)$$

Great miracle

???

Matrix $\boldsymbol{C}$ is diagonalizable and has real eigenvalues and $\lambda_1, \ldots, \lambda_n$, Moreover, $\lambda_n = 0$, $\lambda_{n-1} = 1$, and $\lambda_i > 1$ for $i = 1, \ldots, n-2$.

Recall:

$$C_{ij} = -\frac{m_j}{\mu |x_i - x_j|^3}, \tag{4}$$

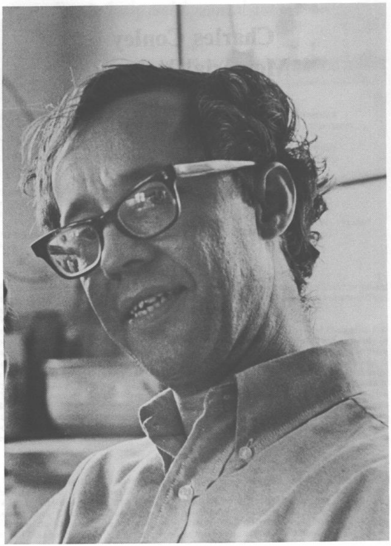
but values of x_i are unknown!

We take $\lambda = \lambda_i > 1$ and the corresponding equation

$$f(\nu)\left(\boldsymbol{X}'' + 2\boldsymbol{J}\boldsymbol{X}'\right) = \boldsymbol{G}\boldsymbol{X}, \qquad \boldsymbol{G} = \operatorname{diag}(3 + 2\delta, -\delta),$$

where $\delta = \lambda - 1 > 0$.

Charles L. Conlay



Charles C. Conley 1933–1984

4×4 **system is not easy.**

$$\boldsymbol{z}' = \boldsymbol{B}(\nu)\boldsymbol{z}, \qquad \boldsymbol{B}(\nu) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_2 \\ \dfrac{1}{f(\nu)}\boldsymbol{G} & -2\boldsymbol{J} \end{bmatrix}, \tag{5}$$

where $\boldsymbol{z} = (\boldsymbol{X}, \boldsymbol{V}) \in \mathbb{C}^4$.

Case $e \neq 0$. The second splitting.

Particular solution

$$\rho(\nu) = \frac{c^2/\mu}{1 + e\cos\nu}, \qquad 2hc^2 = \mu^2(e^2 - 1). \tag{6}$$

Splitting transformation $\boldsymbol{z} \mapsto \boldsymbol{T}(\nu)\boldsymbol{z}$ with matrix

$$\boldsymbol{T}(\nu) := \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{I}_2 \\ \boldsymbol{B}_+(\nu) & \boldsymbol{B}_-(\nu) \end{bmatrix}, \qquad \det \boldsymbol{T}(\nu) = \frac{\Delta^2}{4f(\nu)^2 g(\nu)^2},$$

we get direct product of two systems

$$\boldsymbol{u}' = \boldsymbol{B}_\pm(\nu)\boldsymbol{u}, \qquad \boldsymbol{u} = (u_1, u_2).$$

Wow! We know how to finish the proof!

Remaining case $e = 0$

Real solution $\rho(\nu) = \textit{const}$ (a circular orbit) gives nothing.

But, there is T. Combot a complex orbit

$$f(\nu) = 1 + \mathrm{e}^{\mathrm{i}\nu}.$$

Moreover, we have also similar splitting as for $e > 0$. So, we know how to finish the proof. THE END.